UNITARNI PROSTORI

V je vektorski prostor nad poljem F;
Uvodi se jedna f-ja $\langle \cdot, \cdot \rangle$ $\underline{V^2 \to F}$ koja je skalarni proizvod ako je:

1) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

2) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ $\quad \langle x, y \rangle = \langle y, x \rangle$

4) $\langle x, x \rangle \geq 0$

5) $\langle x, x \rangle = 0 \iff x = 0$

Za 2 nenulta vektora kažemo da su ortogonalni ako je $\langle x, y \rangle = 0$

za $\forall x, y, z \in V$ i $\forall \alpha \in F$

Obično je $F = \mathbb{R}$ ili $F = \mathbb{C}$.

Kada je $F = \mathbb{R}$ 3) $\langle x, y \rangle = \langle y, x \rangle$

Vektorski p. V snadbjeven skalarnim proizvodom zovemo <u>unitarni prostor</u>.

$\|\cdot\| : V \to \mathbb{R}_0^+$ je norma ako ispunjava uslove:

1) $\|x\| \geq 0$ $\qquad$ / $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ /

2) $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$

3) $\|x+y\| \leq \|x\| + \|y\|$ - poznata nejednakost ~~trougla~~

za $\forall x, y \in V$, $\forall \alpha \in \mathbb{R}$.

Vektorski p. snadbjeven normom zovemo <u>normirani prostor</u>.

Norma se može uvesti nezavisno od skalarnog p. a može i preko skalarnog p. na sljedeći način:

$\|x\| = \sqrt{\langle x, x \rangle}$ $\qquad \|x\| = 1$ - normiran $\sqrt{\langle x, y \rangle} = 1$

Udaljenost 2 vektora se definiše sa $d(x, y) = \|x - y\|$

66. Neka je $C[a,b]$ v.p. svih neprekidnih realnih f-ja na $[a,b]$. a) Dokazati da je sa:

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

definisan jedan skalarni p. na v.p. $C[a,b]$

b) Da li su ortogonalni vektori $\cos x$ i $\sin 2x$ u $C[-\pi,\pi]$

c) Naći normu $\|\cos x\|$ u $C[-\pi,\pi]$

d) Naći udaljenost vektora $\sin x$ i $\cos x$ u $C[-\pi,\pi]$

a) 1) $$\langle \alpha f,g\rangle = \int_a^b \alpha f(x)\cdot g(x)\,dx = \int_a^b (\alpha\cdot f(x))\cdot g(x)\,dx = \alpha\int_a^b f(x)\cdot g(x)\,dx$$

$$= \alpha\cdot\langle f,g\rangle$$

2) $$\langle f+g,h\rangle = \int_a^b (f+g)(x)\cdot h(x)\,dx = \int_a^b (f(x)+g(x))\cdot h(x)\,dx =$$

$$= \int_a^b f(x)\cdot h(x)\,dx + \int_a^b g(x)h(x)\,dx = \langle f,h\rangle + \langle g,h\rangle$$

3) $$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g,h\rangle$$

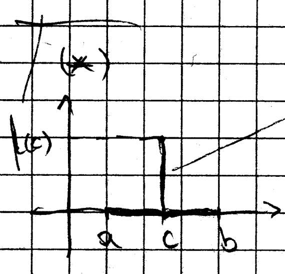
podrazumijeva se da je nad poljem $\mathbb{R}$ pa ne treba $\overline{\langle g,h\rangle}$

4) $$\langle f,f\rangle = \int_a^b f(x)f(x)\,dx = \int_a^b [f(x)]^2\,dx \geq 0$$

5) $$\langle f,f\rangle = 0 \Rightarrow \int_a^b [f(x)]^2\,dx = 0 \overset{(*)}{\Rightarrow} f(x)=0 \quad \forall x\in[a,b]$$

$$\Rightarrow f\equiv 0 \text{ (nula f-ja)}$$

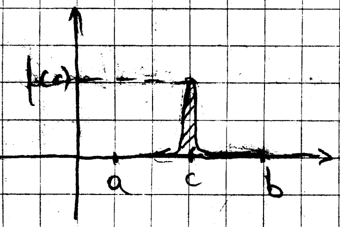
(*)



$\int_a^b f(x)dx=0$, ali $f(x)$ je neprekidna;

Tvrdimo da ako je $\int f(x)dx=0 \Rightarrow f(x)=0$ za $\forall x$

Pretp. da je $f(c)>0$, $c\in(a,b)$



$$\Rightarrow \int_a^b f(x)\,dx > 0$$

PREMAT.: Ako je f-ja f nepr. na [a,b] i $f(x) \geq 0$ $\forall x \in [a,b]$ i $f(c) > 0$, $c \in [a,$ tada je $f(x) > 0$ u nekoj okolini tačke c.

Ovo bi za posljedicu imalo da je $\int_a^b f(x)dx > 0$

$$\Rightarrow f(x) = 0 \quad \text{za } \forall x \in [a,b]$$

Ovo je skalarni proizvod.

b) $\forall x, y \in V;\ x \perp y \Leftrightarrow \langle x, y \rangle = 0$

$$\langle \cos x, \sin 2x \rangle = \int_{-\pi}^{\pi} \cos x \cdot \sin 2x \, dx = 2 \int_{-\pi}^{\pi} \cos^2 x \sin x \, dx =$$

$$= \left| \begin{array}{l} \cos x = t \\ x = -\pi \quad t = -1 \\ x = \pi \quad t = -1 \\ -\sin x\, dx = dt \end{array} \right| = -2 \cdot \int_{-1}^{-1} t^2 dt = 0$$

$$\Rightarrow \cos x \perp \sin 2x$$

c) $\|\cos x\| = \sqrt{\langle \cos x, \cos x \rangle} \overset{*}{=}$

$$\langle \cos x, \cos x \rangle = \int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos 2x) dx =$$

$$= \frac{1}{2} x \Big|_{-\pi}^{\pi} + \underbrace{\frac{1}{2} \cdot \frac{1}{2} \sin 2x \Big|_{-\pi}^{\pi}}_{0} = \frac{1}{2}(\pi + \pi) = \pi$$

$$\overset{*}{=} \sqrt{\pi}$$

d) $\forall\ d(x, y) = \|x - y\|$

$$d(\sin x, \cos x) = \|\sin x - \cos x\| = \sqrt{\langle \sin x - \cos x, \sin x - \cos x \rangle} \overset{*}{=}$$

$$\langle \sin x - \cos x, \sin x - \cos x \rangle = \int_{-\pi}^{\pi} (\sin x - \cos x)^2 dx =$$

$$= \int_{-\pi}^{\pi} (1 - 2 \sin x \cos x) dx = x \Big|_{-\pi}^{\pi} + \frac{1}{2} \cos 2x \Big|_{-\pi}^{\pi} = \pi + \pi + \frac{1}{2}(1 - 1) = 2\pi$$

$$\overset{*}{=} \sqrt{2\pi}$$

67. Dokazati da je sa $\langle x,y\rangle = 2x_1y_1 + 5x_2y_2 + 7x_3y_3$ definisan jedan skalarni p. u $\mathbb{R}^3$. S obzirom na taj skalarni p. ortonormirati ovaj skup vektora: $\{(1,0,1),(1,-1,1),(0,1,1)\}$.

$\alpha \in \mathbb{R}$;

1) $\langle \alpha x, y\rangle = \langle \alpha(x_1,x_2,x_3),(y_1,y_2,y_3)\rangle =$

$= \langle(\alpha x_1, \alpha x_2, \alpha x_3),(y_1,y_2,y_3)\rangle =$

$= 2\cdot \alpha x_1 \cdot y_1 + 5\cdot \alpha x_2 \cdot y_2 + 7\alpha x_3 y_3 =$

$= \alpha(2x_1y_1 + 5x_2y_2 + 7x_3y_3) = \alpha \cdot \langle x,y\rangle$

2) $\langle x+y, z\rangle =$

$= \langle x,z\rangle + \langle y,z\rangle$

3) $\langle x,y\rangle =$

$= \langle y,x\rangle$

4) $\langle x,x\rangle = 2x_1^2 + 5x_2^2 + 7x_3^2 \geq 0$

5) $\langle x,x\rangle = 0 \iff 2x_1^2 + 5x_2^2 + 7x_3^2 = 0 \iff x_1 = x_2 = x_3 = 0$

$\iff x = (0,0,0)$

Gram-Šmitov postupak ortogonalizacije

npr. treba ort. vektore $\{v_1, v_2, \dots v_n\}$ - lin. nez.

$u_1 = v_1$

$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2}\cdot u_1 \quad / \cdot u_1 \qquad \langle u_2, u_1\rangle = \langle v_2, u_1\rangle - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2}\cdot \langle u_1, u_1\rangle = 0$

$u_3 = v_3 - \frac{\langle v_3, u_2\rangle}{\|u_2\|^2}\cdot u_2 - \frac{\langle v_3, u_1\rangle}{\|u_1\|^2}\cdot u_1$

$\vdots$

$u_n = v_n - \frac{\langle v_n, u_{n-1}\rangle}{\|u_{n-1}\|^2}\cdot u_{n-1} - \dots - \frac{\langle v_n, u_1\rangle}{\|u_1\|^2}\cdot u_1$

$\Rightarrow \{u_1, u_2, \dots u_n\}$ ortogonalni

$\left\|\frac{x}{\|x\|}\right\| = \left|\frac{1}{\|x\|}\right|\cdot\|x\| = 1 \rightarrow$ ovako se ortonormiraju vektori

$\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \dots, \frac{u_n}{\|u_n\|}\right\}$ je ortonormirani

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$u_1 = v_1 = (1, 0, 1)$

$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2}\cdot u_1 \overset{*}{=}$

$\langle v_2, u_1\rangle = \langle (1,-1,1), (1,0,1)\rangle = 2\cdot 1 + 5\cdot 0 + 7\cdot 1 = 9$

$\|u_1\|^2 = \langle u_1, u_1\rangle = 9$

$\overset{*}{=} (1,-1,1) - \frac{9}{9}(1,0,1) = (0,-1,0)$

$u_3 = v_3 - \frac{\langle v_3, u_2\rangle}{\|u_2\|^2}\cdot u_2 - \frac{\langle v_3, u_1\rangle}{\|u_1\|^2}\cdot u_1 = \dots = \left(-\frac{7}{9}, 0, \frac{2}{9}\right)$

$\Rightarrow \left\{(1,0,1)(0,-1,0), \left(\frac{-7}{9}, 0, \frac{2}{9}\right)\right\}$ treba ih još podijeliti sa normom svakog posebno

$\|u_1\| = \sqrt{9} = 3$

$\|u_2\| = \sqrt{5}$

$\|u_3\| = \frac{\sqrt{14}}{3}$

$\left\{ \frac{1}{3}(1,0,1), \frac{1}{\sqrt{5}}(0,-1,0), \frac{3}{\sqrt{14}}\left(-\frac{7}{9},0,\frac{2}{9}\right) \right\} \to$ ortonormirani vektori

68. Na skupu $P_2[x]$ (realnih polinoma deg ≤ 2) definisano je preslikavanje:

$\langle f,g \rangle = f(0)\cdot g(0) + \frac{1}{2} f'(0)\cdot g'(0) + \frac{1}{4} f''(0)\cdot g''(0)$

Pokazati da je ovo presl. jedan skalarni proizvod na $P_2[x]$, zatim naći ortonormiranu bazu podprostora generisanog sa vektorima $\{1,x,x^2\}$.

1) $\langle \lambda f, g \rangle = \dots = \lambda \langle f,g \rangle$

2) $\langle f+g, h \rangle = \dots = \langle f,h \rangle + \langle g,h \rangle$

3) $\langle f,g \rangle = \langle g,h \rangle$

4) $\langle f,f \rangle = [f(0)]^2 + \frac{1}{2}[f'(0)]^2 + \frac{1}{4}[f''(0)]^2 \geq 0$

5) $\langle f,f \rangle = 0 \iff [f(0)]^2 + \frac{1}{2}[f'(0)]^2 + \frac{1}{4}[f''(0)]^2 = 0$

$\iff f(0) = f'(0) = f''(0) = 0$

$f(x) = a_2x^2 + a_1x + a_0;$ $\left.\begin{array}{l} f(0)=0: \quad a_0 = 0 \\ f'(0)=0: \quad a_1 = 0 \\ f''(0)=0: \quad a_2 = 0 \end{array}\right\} a_0 = a_1 = a_2 = 0$

$\Rightarrow f(x) \equiv 0 \to$ nula polinom

$$\{1, x, x^2\}$$
$$\quad \| \quad \| \quad \|$$
$$\quad v_1 \quad v_2 \quad v_3$$

$u_1 = 1$

$$u_2 = v_2 - \underbrace{\frac{\langle v_2, u_1 \rangle}{\|u_1\|^2}}_{0} \cdot u_1 = v_2$$

$$\langle v_2, u_1 \rangle = \langle x, 1 \rangle = x|_0 \cdot 1|_0 + \frac{1}{2} \cdot 1 \cdot 0 + \frac{1}{4} \cdot 0 \cdot 0 = 0$$

$x' = 1 \quad 1' = 0$

$x'' = 0, \quad 1'' = 0$

$u_3 = \ldots = v_3$

$$\langle 1, 1 \rangle = 1 \cdot 1 + \frac{1}{2} \cdot 0 \cdot 0 + \frac{1}{4} \cdot 0 \cdot 0 = 1$$

$$\|1\| = \sqrt{\langle 1, 1 \rangle} = 1$$

$$\langle x, x \rangle = 0 \cdot 0 + \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{4} \cdot 0 \cdot 0 = \frac{1}{2}$$

$$\|x\| = \frac{\sqrt{2}}{2}$$

$$\langle x^2, x^2 \rangle = 0 \cdot 0 + \frac{1}{2} \cdot 0 \cdot 0 + \frac{1}{4} \cdot 2 \cdot 2 = 1$$

$$\|x^2\| = 1$$

$$\{1, \sqrt{2} \cdot x, x^2\}$$

69. Dokazati da je sa $\langle u,v\rangle = z_1\cdot\overline{w_1} + (1+i)\cdot z_1\overline{w_2} + (1-i)z_2\cdot\overline{w_1} +$

$$+\,3z_2\overline{w_2}$$

gdje je $u=(z_1,z_2)$, $v=(w_1,w_2)$, definisan i skalarni proizvod nad $\mathbb{C}^2$.

b) Naći normu vektora $u=(1-2i,2+3i)$ gdje je norma indukovana gornjim skalarnim proizvodom.

Neka je $\alpha\in\mathbb{C}$, $a=(a_1,a_2)$, $b=(b_1,b_2)$, $c=(c_1,c_2)\in\mathbb{C}^2$

1) $\langle\alpha a,b\rangle = \alpha a_1\,\alpha a_2 \ldots$

$$=\alpha\langle a,b\rangle$$

2) $\langle a+b,c\rangle = \ldots = \langle a,b\rangle + \langle a,c\rangle$

3) $\langle a,b\rangle = \overline{\langle b,a\rangle}$

$$\overline{\langle b,a\rangle} = \overline{b_1\cdot\overline{a_1} + (1+i)b_1\cdot\overline{a_2} + (1-i)\,b_2\cdot\overline{a_1} + 3b_2\overline{a_2}}$$

$$\overline{z_1\pm z_2} = \overline{z_1}\pm\overline{z_2}$$

$$\overline{z_1\cdot z_2} = \overline{z_1}\cdot\overline{z_2}$$

$$\overline{\overline{z}} = z$$

$$\overline{\langle b,a\rangle} = a_1\overline{b_1} + (1-i)\,a_2\overline{b_1} + (1+i)\,a_1\overline{b_2} + 3a_2\overline{b_2}$$

$$=\langle a,b\rangle$$

4) $\langle a,a\rangle \geq 0$?

$$\langle a,a\rangle = a_1\cdot\overline{a_1} + (1+i)a_1\overline{a_2} + (1-i)\,a_2\overline{a_1} + 3a_2\overline{a_2} \overset{*}{=}$$

$$a=(a_1,a_2)=(x_1+y_1 i,\ x_2+y_2 i)$$

$$\overset{*}{=} \ldots = x_1^2 + y_1^2 + 2\underbrace{x_1x_2}_{x_1x_2+x_2x_1} + 2y_1y_2 - 2x_2y_1 + 2x_1y_2 + 3x_2^2 + 3y_2^2$$

$$= Q(x_1,y_1,x_2,y_2)$$

Treba dokazati da je:

$$Q(x_1,y_1,x_2,y_2)\geq 0 \quad \forall x_1,y_1,x_2,y_2\in\mathbb{R}\ (\forall a\in\mathbb{C}^2)$$

$Q(x_1, y_1, x_2, y_2)$

Kada je ova kvadratna forma pozitivno definitna?

matrica kvadratne forme Q je: (uvijek simetrična)

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

$x_1 x_2$ imamo 0

$x_1 y_2$ imamo 2, prepolovimo $\Rightarrow$ koeficijenti a_{14} i a_{41}

$y_1 y_2$ imamo 2

Napomena:

Matrica kvadratne forme $K(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

je $\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$

$K = (x_1, x_2, x_3) = a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{22}x_2^2 + a_{33}x_3^2$

mat. je: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

Pozitivnu definitnost kvadratne forme možemo utvrditi Silvestrovim kriterijem:

Potražimo glavne minore 1. reda, 2., 3., ...

$$|1| = 1 > 0$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 1 > 0$$

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{vmatrix} = \ldots = 1 > 0$$

Svi glavni minori su pozitivni pa je kvadratna forma pozitivno definitna. (>0)

$$\Rightarrow \langle a,a \rangle = \ldots = Q(x_1, y_1, x_2, y_2) > 0 \quad (\forall a \in \mathbb{C}^2)$$

$$\left(\forall (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \setminus \{0,0,0,0\}\right)$$

Za $(x_1, y_1, x_2, y_2) = (0,0,0,0)$ imamo:

$$\langle (0,0), (0,0) \rangle = 0$$

5) $\langle a,a \rangle = 0 \Leftrightarrow a = (0,0)$

Ovo smo dokazali u prethodnom dijelu.

(pozitivno je za sve osim za 0 kada je 0)

Norma vektora:

$$\|(1-2i, 2+3i)\| = \sqrt{\langle(1-2i, 2+3i), (1-2i, 2+3i)\rangle} = \ldots = \sqrt{50} = 5\sqrt{2}$$

70. Na skupu $P_2[x]$ polinoma stepena ≤ 2, definisano je preslikavanje:

$$\langle f, g\rangle = 2f(-1)g(-1) + f(0)g(0) + 2f(1)\cdot g(1)$$

Dokazati da je ovo presl. jedan skalarni proizvod na $P_2[x]$. Naći zatim ortonormiranu bazu podprostora generisanim vektorima: $e_1 = 1-x$, $e_2 = x^2-2$.

5) $\langle f, f\rangle = 0 \iff f = 0$ (0 - nula pol.)

$$2(f(-1))^2 + (f(0))^2 + 2(f(1))^2 = 0 \Rightarrow f(-1) = f(0) = f(1) = 0$$

$$\Rightarrow f \equiv 0$$

Napomena: Ako polinom najviše n-tog stepena ima $n+1$ različitu nulu, onda on mora biti nula polinom.

$$\left\{ -\frac{1}{3}x + \frac{1}{3};\ \frac{1}{\sqrt{5}}x^2 - \frac{2}{3\sqrt{5}}x - \frac{4}{3\sqrt{5}} \right\}$$

71. Neka je M_m^n vektorski prostor mat. $m \times n$.

a) Dokazati da je sa $\langle A, B\rangle = tr(B^T \cdot A)$ jedan skalarni proizvod u M_m^n.

b) Da li je skup $\left\{ \overset{E_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{E_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{E_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{E_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$ jedna ortonormirana baza u.p. M_2^2 s obzirom na dati skalarni proizvod

$\alpha \in \mathbb{R}, \quad A, B, C \in M_{mn}^{m}$

$$\langle A, B\rangle = tr\begin{pmatrix} b_{11} & b_{21} & \dots & b_{m1} \\ b_{12} & b_{22} & \dots & b_{m2} \\ \vdots \\ b_{1m} & b_{2m} & \dots & b_{mm}\end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm}\end{pmatrix} =$$

$$= tr\begin{pmatrix} \sum_{i=1}^{m} b_{i1}a_{i1} & & \\ & \sum_{i=1}^{m} b_{i2}a_{i2} & \\ & & \ddots \\ & & & \sum_{i=1}^{m} b_{im}a_{im}\end{pmatrix} = \sum_{j=1}^{m}\sum_{i=1}^{m} b_{ij}a_{ij}$$

1) $\langle \alpha A, B\rangle = \dots = \alpha \langle A, B\rangle$

2) $\langle A+B, C\rangle = tr\left(C^T \cdot (A+B)\right) = \sum_{j=1}^{m}\sum_{i=1}^{m} c_{ij}(a_{ij}+b_{ij}) = \sum_{j=1}^{m}\sum_{i=1}^{m}(c_{ij}a_{ij}+c_{ij}b_{ij})$

$$= \sum_{j=1}^{m}\sum_{i=1}^{m} c_{ij}a_{ij} + \sum_{j=1}^{m}\sum_{i=1}^{m} c_{ij}b_{ij}$$

$$= tr(C^T \cdot A) + tr(C^T \cdot B) = \langle A, C\rangle + \langle B, C\rangle$$

3) $\langle A, B\rangle = \dots = \langle B, A\rangle$

4) $\langle A, A\rangle = \dots = \sum_{j=1}^{m}\sum_{i=1}^{m} a_{ij}^2 \geq 0$ (suma kvadrata ≥ 0)

5) $\langle A, A\rangle = 0 \Rightarrow \sum_{j=1}^{m}\sum_{i=1}^{m} a_{ij}^2 = 0 \Leftrightarrow a_{ij} = 0$ za $\forall i \in \{1, \dots, m\}$
$\forall j \in \{1, \dots m\}$

$\Leftrightarrow A = 0$

$\Rightarrow$ ovo je skalarni proizvod.

b) $\|E_1\| = \left\|\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right\| = \sqrt{\left\langle\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right\rangle} = \sqrt{tr\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)} = \sqrt{tr\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}} = \sqrt{1} = 1$

Slično: $\|E_2\| = \|E_3\| = \|E_4\| = 1$

$\langle E_1, E_2\rangle = tr\left(E_2^T \cdot E_1\right) = tr\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = tr\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = 0 \Rightarrow E_1 \perp E_2$

Slično: $E_1 \perp E_3$, $E_1 \perp E_4$, $E_2 \perp E_3$, $E_2 \perp E_4$, $E_3 \perp E_4$

72. S obzirom na skalarni proizvod $\langle A, B\rangle = tr(B^T \cdot A)$ naći ortonormiranu bazu podprostora $M_2(\mathbb{R})$ od v.p. $\mathbb{R}$, generisanog vektorima:

$v_1 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 & 5 \\ 2 & -3 \end{pmatrix}$, $v_3 = \begin{pmatrix} -7 & 0 \\ 29 & 7 \end{pmatrix}$

$u_1 = v_1$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} \cdot u_1 = \dots = \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}$$

$$u_3 = v_3 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} \cdot u_2 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} \cdot u_1 = \dots = \begin{pmatrix} 0 & -10 \\ 25 & 0 \end{pmatrix}$$

$\{u_1, u_2, u_3\}$ je ortogonalna
treba je još ortonormirati:

$\|u_1\| = \sqrt{8}$

$\|u_2\| = \sqrt{29}$

$\|u_3\| = \sqrt{725} = 5\sqrt{29}$

$$\left\{ \frac{1}{\sqrt{8}} \cdot \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \frac{1}{\sqrt{29}} \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}, \frac{1}{\sqrt{29}} \begin{pmatrix} 0 & -2 \\ 5 & 0 \end{pmatrix} \right\}$$

73. Za koje ~~vr~~vrijednosti realnog parametra λ je sa $\langle x, y\rangle = 4x_1y_1 - 2x_1y_2 - 2x_2y_1 + \lambda x_2y_2$ definisan jedan skalarni proizvod u prostoru $\mathbb{R}^2$? Za $\lambda = 3$ naći normu vektora $(-2, 3)$, te ispitati ortogonalnost vektora $(-1, \sqrt{2})$ $(\sqrt{2}, 2)$

$x, y, z \in \mathbb{R}^2$

$x = (x_1, x_2)$

$y = (y_1, y_2)$

$z = (z_1, z_2)$

1) $\langle \alpha x, y\rangle = \dots = \alpha \cdot \langle x, y\rangle$

2) $\langle x + y, z\rangle = \dots = \langle x, z\rangle + \langle y, z\rangle$

3) $\langle x, y\rangle = \dots = \langle y, x\rangle$

4) $\langle x, x\rangle = \langle (x_1, x_2)(x_1, x_2)\rangle = 4x_1^2 - 2x_1x_2 - 2x_2x_1 + \lambda x_2^2 =$

$= 4x_1^2 - 4x_1x_2 + \lambda x_2^2 =$

$= (2x_1 - x_2)^2 - x_2^2 + \lambda x_2^2$

$= (2x_1 - x_2)^2 + (\lambda - 1)\, x_2^2 \geq 0 \quad$ za $\left(\forall x \in \mathbb{R}^2 \quad \text{tj. } \forall x_1, x_2 \in \mathbb{R}\right) \Rightarrow \underline{\lambda \geq 1}$

5) $\langle x, x \rangle = 0 \iff (2x_1 - x_2)^2 + (\lambda - 1)\, x_2^2 = 0$ želimo da bude $= 0$ kada je $x_1 = x_2 = 0$

$\iff x_1 = x_2 = 0 \quad \Rightarrow \lambda > 1$!

Naime, za $\lambda = 1$ ovaj bi izraz mogao biti $= 0$ za npr $x_1 = 1$, $x_2 = 2$! Pa $\lambda \neq 1$ jer ne bi važila 5. aksioma tj. $\lambda > 1$.

$\lambda = 3 \qquad \|(-2, 3)\| = \ldots = \sqrt{67}$

$\langle (-1, \sqrt{2}), (\sqrt{2}, 2) \rangle = \ldots = 2\sqrt{2} \neq 0 \rightarrow$ nisu ortogonalni.

1. Alg. strukture ili dokazati da je neki skup v.p.
2. Baza, dimenzija, linearna nez. vektora
3. Lin. operatora
4. Unitarni prostori